[redacted]

Office Hours: 1:30-3:00 in COM B-006

7.1 Integration by Parts *(continued)*

Summary: $\int u\,dv = uv - \int v\,du$

1. Pick u = ??. The rest is dv.
2. Compute du and v.

Entry Task: Evaluate

1. $\int \frac{\ln(x)}{\sqrt{x}}\,dx$

$u = \ln(x) \qquad dv = x^{-1/2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = 2x^{1/2} = 2\sqrt{x}$

$= 2\sqrt{x}\ln(x) - \int 2\sqrt{x}\,\frac{1}{x}\,dx$

$x^{-1/2}$

$= 2\sqrt{x}\ln(x) - 2(2\cdot x^{1/2}) + C$

$= \boxed{\underbrace{2\sqrt{x}}_{F}\,\underbrace{\ln(x)}_{S} - 4\sqrt{x} + C}$

CHECK

$F'S \quad F S'$

$\frac{1}{\sqrt{x}}\ln(x) + 2\sqrt{x}\,\frac{1}{x} - \frac{2}{\sqrt{x}}$ ✓

2. $\int_0^1 x^2 e^{x/3}\,dx$

$u = x^2 \qquad dv = e^{\frac{1}{3}x}\,dx$

$du = 2x\,dx \qquad v = 3e^{1/3 x}$

$= 3x^2 e^{\frac{1}{3}x}\Big|_0^1 - \int_0^1 6x e^{1/3 x}\,dx$

$= 3e^{1/3} - \left[18xe^{1/3 x}\Big|_0^1 - \int_0^1 18e^{1/3 x}\,dx\right]$

$u = 6x \qquad dv = e^{1/3 x}\,dx$

$du = 6\,dx \qquad v = 3e^{1/3 x}$

$= 3e^{1/3} - 18e^{1/3} + 18\cdot 3e^{1/3 x}\Big|_0^1$

$= -15e^{1/3} + 54(e^{1/3} - e^{0})$

$= \boxed{39e^{1/3} - 54}$

Integration by parts is good for:

Products: $\underset{u}{x}e^{x}, \underset{u}{x^2}\cos(3x), \underset{u}{x}\sin(5x)$

Logs: $\underset{u}{\ln(x)}, x^{10}\underset{u}{\ln(x)}, \frac{\overset{u}{\ln(x)}}{x^3}, \ldots$

Inv. Tri: $\underset{u}{\sin^{-1}(x)}, \mathrm{x}\underset{u}{\tan^{-1}(x)}, \ldots$

Products: $\underset{u}{e^x}\sin(x), \underset{u}{e^x}\cos(x)$

Example:

$$\int \cancel{\sin^{-1}(x)}\,dx = \int \cancel{\arcsin(x)}\,dx$$

$\tan^{-1}(x)$ $\qquad$ $\arctan(x)$

$u = \tan^{-1}(x)$ $\quad dv = dx$

$du = \frac{1}{1+x^2}dx$ $\quad v = x$

$$= x\tan^{-1}(x) - \int \frac{x}{1+x^2}\,dx$$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= x\tan^{-1}(x) - \int \frac{1}{u}\cdot\frac{1}{2}\,du$$

$$= x\tan^{-1}(x) - \frac{1}{2}\ln|u| + C$$

$$= \boxed{x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C}$$

ASIDE

$y = \tan^{-1}(x)$

$\Rightarrow \tan(y) = x$

$\Rightarrow \sec^2(y)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2(y)}$

$= \frac{1}{1+\tan^2(y)}$

$= \frac{1}{1+x^2}$

Example: (Never ending integration by parts and how to end it):

$$\int e^x \cos(x)\, dx$$

$u = e^x \quad dv = \cos(x)dx$

$du = e^x dx \quad v = \sin(x)$

$e^x \sin(x) - \int e^x \sin(x) dx$

$u = e^x \quad dv = \sin(x) dx$

$du = e^x dx \quad v = -\cos(x)$

$e^x \sin(x) - \left(-e^x \cos(x) - \int e^x (-\cos(x)) dx \right)$

$\Rightarrow \int e^x \cos(x) dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) dx$

$2 \int e^x \cos(x) dx = e^x \sin(x) + e^x \cos(x) + C$

$\int e^x \cos(x) dx = \frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + \frac{C}{2}$

7.2 Trigonometric Integral Methods

Goal: A procedure to integrate *any* combination of trig functions.

Motivating examples: ***These are substitution problems, what is u?***

$$\int \underbrace{\sin^3(x)}_{u}\,(1-\underbrace{\sin^2(x)}_{u})\,\underbrace{\cos(x)\,dx}_{du}$$

$$\int (1-\underbrace{\cos^2(x)}_{u})\,\underbrace{\cos^5(x)}_{u}\,\underbrace{\sin(x)\,dx}_{-du}$$

$$\int \underbrace{\tan^5(x)}_{u}\,(1+\underbrace{\tan^2(x)}_{u})\,\underbrace{\sec^2(x)\,dx}_{du}$$

$$\int \underbrace{\sec^6(x)}_{u}\,\underbrace{\sec(x)\tan(x)\,dx}_{du}$$

7.2 idea: Use trig identities to turn almost all trig problems into one of these situations!

Essential Tools

See my online postings (or the Appendix of your book) for a more general discussion and proofs of trig identities.

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \cot(x) = \frac{\cos(x)}{\sin(x)},$$
$$\sec(x) = \frac{1}{\cos(x)}, \csc(x) = \frac{1}{\sin(x)}.$$

$$\sin^2(x) + \cos^2(x) = 1$$
$$\tan^2(x) + 1 = \sec^2(x)$$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$
$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$
$$\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$$

Case 1 (cosine or sine has an odd power)

$i) \int \sin^2(x)\cos^3(x)\,dx$ ← ODD

PULL OUT ONE COSINE

$$\int \sin^2(x)\cos^2(x)\cos(x)\,dx$$

USE IDENTITY

$$\int \sin^2(x)(1-\sin^2(x))\cos(x)\,dx$$

$u = \sin(x)$

$du = \cos(x)\,dx$

$$\int u^2(1-u^2)\,du$$

$$\int u^2 - u^4\,du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$$

$$\boxed{= \frac{1}{3}\sin^3(x) - \frac{1}{5}\sin^5(x) + C}$$

$ii) \int \sin^3(x)\,dx$ ← ODD

PULL OUT ONE SINE

$$\int \sin^2(x)\sin(x)\,dx$$

USE IDENTITY

$$\int (1-\cos^2(x))\sin(x)\,dx$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$-\frac{1}{\sin(x)}du = dx$

$$\int 1-u^2 \cdot \sin(x) \cdot \frac{-1}{\sin(x)}\,du$$

$$-\int 1-u^2\,du$$

$$-\left(u - \frac{1}{3}u^3\right) + C$$

$$\boxed{-\cos(x) + \frac{1}{3}\cos^3(x) + C}$$

Case 2 (Both have even powers)

$i) \int \cos^2(x)\,dx$ ← even

$$\int \frac{1}{2}(1+\cos(2x))\,dx$$

$$\frac{1}{2}\int 1+\cos(2x)\,dx$$

$$\frac{1}{2}\left(\theta + \frac{1}{2}\sin(2x)\right) + C$$

$$\boxed{\frac{1}{2}\theta + \frac{1}{4}\sin(2x) + C}$$

$ii) \int \sin^4(x)\,dx$

$$\int \sin^2(x)\sin^2(x)\,dx$$

$$= \int \frac{1}{2}(1-\cos(2\theta))\frac{1}{2}(1-\cos(2\theta))\,d\theta$$

$$= \frac{1}{4}\int 1-2\cos(2\theta)+\cos^2(2\theta)\,d\theta$$

$$= \frac{1}{4}\left[\theta - \sin(2\theta) + \int \frac{1}{2}(1+\cos(4\theta))\,d\theta\right]$$

$$= \frac{1}{4}\theta - \frac{1}{4}\sin(2\theta) + \frac{1}{8}\left(\theta + \frac{1}{4}\sin(4\theta)\right) + C$$

$$= \boxed{\frac{3}{8}\theta - \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta) + C}$$

Case 3 (even power on secant)

$$\int \tan^2(x)\sec^4(x)\,dx$$

even

PULL OUT $\sec^2(x)$

$$\int \tan^2(x)\sec^2(x)\sec^2(x)\,dx$$

$$\int \tan^2(x)(1+\tan^2(x))\sec^2(x)\,dx$$

$u=\tan(x)$

$du=\sec^2(x)\,dx$

$$\int u^2(1+u^2)\,du$$

$$\int u^2+u^4\,du$$

$$\frac{1}{3}u^3+\frac{1}{5}u^5+C$$

$$\boxed{\frac{1}{3}\tan^3(\theta)+\frac{1}{5}\tan^5(\theta)+C}$$

Case 4 (Odd power on ~~secant~~ tangent, and at least one ~~tangent~~ sec(x))

$$\int \tan^3(x)\sec^5(x)\,dx$$

odd

PULL OUT $\sec(x)\tan(x)$

$$\int \tan^2(x)\sec^4(x)\sec(x)\tan(x)\,dx$$

$$\int (\sec^2(x)-1)\sec^4(x)\sec(x)\tan(x)\,dx$$

$u=\sec(x)$

$du=\sec(x)\tan(x)\,dx$

$$\int (u^2-1)u^4\,du$$

$$\int u^6-u^4\,du$$

$$\frac{1}{7}u^7-\frac{1}{5}u^5+C$$

$$\boxed{\frac{1}{7}\sec^7(x)-\frac{1}{5}\sec^5(x)+C}$$

Notes:
And if you've tried all methods and are stuck, here are things to try:

1. Rewrite in terms of sin(x) and cos(x).
2. Rewrite in terms of sec(x) and tan(x).
3. Try using trig identities.

And there are still a few "holes".
Particularly, odd power on sec(x).
For these you can quote
(proof in the book):

$$\int \tan(x)\,dx = \ln|\sec(x)| + C$$

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(\mathrm{x}) + \tan(\mathrm{x})| + C$$

What is the first step in each integral below?

ODD

$$\int \sin^3(x)\cos^4(x)dx = \int (1-\cos^2(x))\cos^4(x)\sin(x)dx$$

ODD

$$\int \sin^5(x)\cos^3(x)dx = \int \sin^5(x)(1-\sin^2(x))\cos(x)dx$$

EVEN

$$\int \cos^4(x)dx = \int \cos^2(x)\cos^2(x)dx = \int \frac{1}{2}(1+\cos(2x))\frac{1}{2}(1+\cos(2x))dx$$

EVEN

$$\int \tan^5(x)\sec^4(x)dx = \int \tan^5(x)(1+\tan^2(x))\sec^2(x)dx$$

ODD

$$\int \tan^5(x)\sec(x)dx = \int (\sec^2(x)-1)^2\sec(x)\tan(x)dx$$